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HIGH FREQUENCY ASYMPTOTICS OF GLOBAL VIBRATIONS IN A PROBLEM WITH CONCENTRATED MASS

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ABSTRACT. We consider an elastic system containing a small region where the density is very much higher then elsewhere. Such system possesses two types of eigenvibrations, which are local and global vibrations. Complete asymptotic expansions of global eigenvibrations for ordinary differential operator of the fourth order are constructed using WKB – technique.

Keywords: high frequency, concentrated mass, eigenfunction approximation, WKB method, quantization

MSC: 34E20, 74K10, 34L20

Introduction. Heterogeneous systems give rise to new effects that do not reside in the separated system parts. For instance, problems with local density perturbation are characterized by a presence of local and global vibrations. For the first time the effects have been described by E. Sanchez-Palencia [6,7]. Vibrating systems with a local mass perturbation are investigated starting from O. A. Oleinik [3,4]. The effect of local vibrations has been studied for wide class of the systems, for instance see [1,2]. The global vibrations have been remained to be weakly analyzed so far. A complete asymptotic expansions of the global eigenvibrations for one-dimensional system of the forth order with locally perturbed density are constructed in this work.

1. Problem statement. Let a differential expression L be given by

$$Lv = (k_0(x)v'')'' - (k_1(x)v')' + k_2(x)v,$$

where the functions $k_0 > 0$, $k_1, k_2 \geq 0$ are smooth at $[a, b]$. Denote by $[v]_{x=c}$ a jump of a function v at point c . We investigate asymptotic behaviour as $\varepsilon \rightarrow 0$ of eigenvalues λ_ε and eigenfunctions u_ε of the problem

$$Lu_\varepsilon - \lambda_\varepsilon p(x)u_\varepsilon = 0, \quad x \in (a, -\varepsilon) \cup (\varepsilon, b), \quad (1)$$

$$Lu_\varepsilon - \lambda_\varepsilon \varepsilon^{-m} q(x/\varepsilon)u_\varepsilon = 0, \quad x \in (-\varepsilon, \varepsilon), \quad (2)$$

$$u_\varepsilon(a) = u'_\varepsilon(a) = 0, \quad u_\varepsilon(b) = u'_\varepsilon(b) = 0, \quad (3)$$

$$[u_\varepsilon]_{x=\pm\varepsilon} = [u'_\varepsilon]_{x=\pm\varepsilon} = [u''_\varepsilon]_{x=\pm\varepsilon} = [u'''_\varepsilon]_{x=\pm\varepsilon} = 0. \quad (4)$$

For each fixed $\varepsilon > 0$ the problem possesses a countable set of eigenvalues. Bihaviour as $\varepsilon \rightarrow 0$ for each eigenvalue λ^ε and corresponding eigenfunctions u_ε depending on a value of real parameter m is investigated in [1]. In the case $m > 4$ problem (1) – (4) possesses local eigenvibrations¹ with corresponding eigenvalues $\lambda_n^\varepsilon = O(\varepsilon^{m-4})$, $\varepsilon \rightarrow 0$, and with eigenfunctions u_n^ε , which localize itself in the interval of density perturbation $(-\varepsilon, \varepsilon)$, rapidly vanishing outside the interval. Nevertheless, qualitative behaviour of the vibrating system is not yet described completely by local eigenvibrations. Models with concentrated masses possess also global vibrations [7]. As indicated below, the global

¹We use a term "eigenvibration" to denote a pare of eigenvalue and corresponding eigenfunction

vibrations are supported by eigenfunction sequences with nontrivial limits $u_{n(\varepsilon)}^\varepsilon \rightarrow v_0$ for eigenvalues $\lambda_{n(\varepsilon)}^\varepsilon \rightarrow \lambda_0 > 0$ with $n(\varepsilon) \rightarrow \infty$. Dependence $n(\varepsilon)$ is a discrete one that causes construction of asymptotics along certain sequences of the small parameter only. A family of the sequences is bound by a deformation parameter, which is present in the lower terms of the constructed expansions. Asymptotics depend on the value of m . We choose $m = 8$ as a pattern case.

2. Asymptotics of global vibrations: the leading terms. We seek the asymptotic expansions of the eigenvalues λ_ε and the eigenfunctions u_ε of problem (1) – (4) in the form

$$\lambda_\varepsilon \sim \sum_{i=0}^{\infty} \varepsilon^i \lambda_i, \quad (5)$$

$$u_\varepsilon(x) \sim \sum_{i=0}^{\infty} \varepsilon^i v_i(x), \quad x \in (a, -\varepsilon) \cup (\varepsilon, b), \quad v_0 \neq 0. \quad (6)$$

In order to explore the function u_ε in the region $(-\varepsilon, \varepsilon)$, we consider problem (1) – (4) in variables $\xi = \varepsilon^{-1}x$. Taking into account

$$k_i(\varepsilon\xi) = \sum_{j=0}^{\infty} (\varepsilon\xi)^j k_i^{(j)}(0) (j!)^{-1},$$

we transform the differential expression L to $L_\varepsilon = \varepsilon^{-4} \sum_{j=0}^{\infty} \varepsilon^j \mathcal{L}(j)$, where

$$\mathcal{L}(j) = \frac{k_0^{(j)}(0)}{j!} \frac{d^2}{d\xi^2} \xi^j \frac{d^2}{d\xi^2} - \frac{k_1^{(j-2)}(0)}{(j-2)!} \frac{d}{d\xi} \xi^{j-2} \frac{d}{d\xi} + \frac{k_2^{(j-4)}(0)}{(j-4)!} \xi^{j-4}.$$

We use notation $k_n^{(i)}(x)$ for the i -th derivative of a function $k_n(x)$ in case if $i \in \mathbb{N}$, $k_n^{(0)}(x) = k_n(x)$ and $k_n^{(j)}(x) \equiv 0$ if $j < 0$.

Hence, the eigenfunction u_ε , which corresponds to λ^ε , is a solution of the equation with small parameter nearby the highest derivative

$$\varepsilon^8 L_\varepsilon U_\varepsilon - \lambda_\varepsilon q(\xi) U_\varepsilon = 0, \quad U_\varepsilon = U_\varepsilon(\xi), \quad \xi \in (-1, 1). \quad (7)$$

According to the method of WKB-approximations [4], we seek the solution of (7) as linear combination of the series $e^{\varepsilon^{-1}S(\xi)} \sum_{i=0}^{\infty} \varepsilon^i a_i(\xi)$. Equality (7) is guarantied by the choice of functions S and a_i . In particular, the phase function S is a solution of the eikonal equation

$$k_0(0)S'^4 - \lambda_0 q(\xi) = 0, \quad \xi \in (-1, 1). \quad (8)$$

Therefore we fix

$$S(\xi) = (\lambda_0 k_0^{-1}(0))^{1/4} \int_{-1}^{\xi} q^{1/4}(t) dt$$

and, without loss of generality, we derive the eigenfunction u_ε in the form

$$u_\varepsilon(\varepsilon\xi) \stackrel{def}{=} U_\varepsilon(\xi) = \varepsilon^4 \sum_{i=0}^{\infty} \varepsilon^i \langle f_i(\xi), N(\xi, \varepsilon^{-1}S) \rangle, \quad \xi \in (-1, 1), \quad (9)$$

with f_i being a vector-function with values in \mathbb{R}^4 , $\langle \cdot, \cdot \rangle$ being the standard scalar product in \mathbb{R}^4 , and the operator N mapping $[-1, 1] \times C^\infty[-1, 1]$ in \mathbb{R}^4 according to

$$N(\xi, \tau) = (\cos \tau(\xi), \sin \tau(\xi), \exp(-\tau(\xi) + \tau(-1)), \exp(\tau(\xi) - \tau(1))).$$

The choice of the multiplier ε^4 and the argument shifts in exponents of series (9) is related to a certain normalization of the eigenfunction u_ε , which is described in Section 4 (all other eigenfunctions differ by a constant multiplier).

We construct formal asymptotic expansions (5), (6), (9) satisfying all conditions of problem (1) – (4). In particular, equality (1) holds if

$$Lv_0 - \lambda_0 p(x)v_0 = 0, \quad x \in (a, 0) \cup (0, b), \quad (10)$$

$$Lv_i - \lambda_0 p(x)v_i = p(x) \sum_{j=1}^i \lambda_j v_{i-j}, \quad x \in (a, 0) \cup (0, b). \quad (11)$$

Let consider the action of the differential expression L_ε on the function U_ε . We note that $\frac{d}{d\xi} N(\xi, \tau) = \tau'(\xi)TN(\xi, \tau)$, where the matrix

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}, \quad T_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

has the property $T^* = T^3$, and T^4 is a unit matrix. Counting

$$\frac{d}{d\xi} \langle f_k(\xi), N(\xi, \varepsilon^{-1}S) \rangle = \left\langle \left(\varepsilon^{-1}S'(\xi)T^3 + \frac{d}{d\xi} \right) f_k(\xi), N(\xi, \varepsilon^{-1}S) \right\rangle,$$

we get

$$L_\varepsilon U_\varepsilon = \sum_{i=0}^{\infty} \varepsilon^{i-8} \sum_{j=0}^i \left\langle \sum_{l=0}^4 \mathcal{L}_l(j-l)f_{i-j}(\xi), N(\xi, \varepsilon^{-1}S) \right\rangle,$$

where $\mathcal{L}_n(k)$ are the differential expressions of the n -th order with coefficients depending on S' and ξ^k . In particular, $\mathcal{L}_0(i) = k_0^{(i)}(0)(i!)^{-1}S'^4\xi^i$, and

$$\mathcal{L}_1(i) = k_0^{(i)}(0)(i!)^{-1} \left(S'^3\xi^i \frac{d}{d\xi} + S'^2\xi^i \frac{d}{d\xi} S' + S' \frac{d}{d\xi} S'^2\xi^i + \frac{d}{d\xi} S'^3\xi^i \right) T.$$

We recall that the differential expressions $\mathcal{L}_n(t)$, $n = 0, \dots, 4$, are equal to zero one for $t < 0$. Hence, equation (2) yields

$$(\mathcal{L}_0(0) - \lambda_0 q)f_i + (\mathcal{L}_0(1) + \mathcal{L}_1(0) - \lambda_1 q)f_{i-1} = \chi_i,$$

with $\chi_i = - \sum_{j=2}^i \sum_{l=0}^4 (\mathcal{L}_l(j-l) - \lambda_j q)f_{i-j}$ for $i \geq 2$, and $\chi_t \equiv 0$ for $t < 2$. The expression

$\mathcal{L}_0(0) - \lambda_0 q$ is equal to zero because it is the left-hand side of the eikonal equation given by (8). Therefore the coefficients f_i of expansion (9) solve the system of the first-order differential equations

$$(\mathcal{L}_0(1) + \mathcal{L}_1(0) - \lambda_1 q)f_i = \chi_{i+1}, \quad i = 0, 1, 2, \dots \quad (12)$$

Boundary conditions (3) give

$$v_i(a) = v'_i(a) = 0, \quad v_i(b) = v'_i(b) = 0, \quad i = 0, 1, 2, \dots \quad (13)$$

Interfacial conditions (4) applied to (6) and (9) yield

$$\sum_{j=0}^i (\pm 1)^j (j!)^{-1} v_{i-j}^{(j)}(\pm 0) = \langle f_{i-4}(\pm 1), N(\pm 1, \varepsilon^{-1}S) \rangle, \quad (14)$$

$$\sum_{j=0}^i (\pm 1)^j (j!)^{-1} v_{i-j}^{(j+1)}(\pm 0) = \langle (S'T^3 f_{i-2} + f'_{i-3})(\pm 1), N(\pm 1, \varepsilon^{-1}S) \rangle,$$

$$\sum_{j=0}^i (\pm 1)^j (j!)^{-1} v_{i-j}^{(j+2)}(\pm 0) = \langle (S'^2 T^2 f_i + D_i)(\pm 1), N(\pm 1, \varepsilon^{-1}S) \rangle, \quad (15)$$

$$F_i(\pm 0) = \langle (S'^3 T f_i + E_i)(\pm 1), N(\pm 1, \varepsilon^{-1}S) \rangle,$$

where we used notation

$$f_j \equiv 0 \quad \text{for } j < 0;$$

$$D_i = 2S'T^3 f'_{i-1} + S''T^3 f_{i-1} + f''_{i-2} \quad \text{for } i = 0, 1, \dots;$$

F_l take zero values for $l = 0, 1$, and

$$F_i(\pm 0) = \sum_{j=0}^{i-2} (\pm 1)^j (j!)^{-1} v_{i-j-2}^{(j+3)}(\pm 0) \quad \text{for } i = 2, 3, \dots;$$

$$\text{and } E_i = \left(S'^2 \frac{d}{d\xi} + S' \frac{d}{d\xi} S' + \frac{d}{d\xi} S'^2 \right) T^2 f_{j-1} + \left(S' \frac{d^2}{d\xi^2} + \frac{d}{d\xi} S' \frac{d}{d\xi} + \frac{d^2}{d\xi^2} S' \right) \times \\ T^3 f_{j-2} + f_{j-3}'''' \text{ for } i = 0, 1, \dots$$

According to conditions (10), (13), (14) for $i = 0$, the leading terms λ_0 and v_0 of expansions (5), (6) are correspondingly an eigenvalue and eigenfunction of the problem

$$Lv_0 - \lambda_0 p(x) v_0 = 0, \quad x \in (a, 0) \cup (0, b), \\ v_0(a) = v_0'(a) = 0, \quad v_0(0) = v_0'(0) = 0, \quad v_0(b) = v_0'(b) = 0. \quad (16)$$

We restrict ourself to considering only simple eigenvalues of three-point Dirichlet problem (16). Nevertheless, there are situations when all the eigenvalues has multiplicity more then 1 (for instance, $a = -b$ and p is an even function). We fix a simple eigenvalue λ_0 of (16) and the corresponding eigenfunction v_0 such that $v_0(x) \equiv 0$ for $x \in (0, b)$ and

$$\int_a^0 p v_0^2 dx = 1.$$

Equalities (11), (13), (14) for $i = 1$ give

$$Lv_1 - \lambda_0 p(x) v_1 = \lambda_1 p(x) v_0, \quad x \in (a, 0) \cup (0, b), \\ v_1(a) = v_1'(a) = 0, \quad v_1(-0) = 0, \quad v_1'(-0) = v_0''(-0), \\ v_1(+0) = v_0'(0) = 0, \quad v_1(b) = v_1'(b) = 0. \quad (17)$$

Three-point problem (17) has solution if and only if the parameter λ_1 takes value

$$\lambda_1 = k_0(0) v_0''(-0)^2.$$

We fix a solution v_1 of (17) such that $\int_a^0 p v_1 v_0 dx = 0$. One can note that $v_1(x) \equiv 0$ for $x \in (0, b)$.

Conditions (12), (15) for $i = 0$ yield that the leading term f_0 of series (9) satisfies the problem

$$f_0' = A(\xi) f_0, \quad \xi \in (-1, 1), \quad (18)$$

$$\langle T^2 f_0(\pm 1), N(\pm 1, \varepsilon^{-1} S) \rangle = \sigma^\pm, \quad \langle T f_0(\pm 1), N(\pm 1, \varepsilon^{-1} S) \rangle = 0, \quad (19)$$

with $\sigma^+ = 0$, $\sigma^- = S'(-1)^{-2} v_0''(-0)$. The matrix of the linear homogeneous system of the first-order differential equations (18) is

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \quad A_1 = \begin{pmatrix} \eta & \theta \\ -\theta & \eta \end{pmatrix}, \quad A_2 = \begin{pmatrix} \eta - \theta & 0 \\ 0 & \eta + \theta \end{pmatrix},$$

with

$$\eta = -3q'(\xi)(8q(\xi))^{-1}, \quad \theta = 1/4 \left(\lambda_0^{-3} k_0^{-5}(0) q(\xi) \right)^{1/4} (\lambda_1 k_0(0) - \lambda_0 k_0'(0) \xi).$$

On the one hand, problem (18), (19) depends on the small parameter, on another hand, it is a boundary value problem, and hence, it is an ill-posed one. Both of the difficulties can be solved by exploring problem (18), (19) along with similar problems, which appear below, along discrete sequences of the small parameter $\varepsilon_l \rightarrow 0$. Then "along" these sequences the problems have a unique solution, which is independent of ε_l up to the exponentially small terms. In particular, this property is consistant with a discrete character of the high-frequency-vibrations effect, which is under consideration.

Proposition 1. *Let $w : [-1, 1] \rightarrow \mathbb{R}^4$ be a smooth vector-function, and σ be a vector in \mathbb{R}^4 . There exists a small parameter sequence $\{\varepsilon_l\}_{l=1}^\infty$ such that the problem*

$$y'(\xi, \varepsilon) = A(\xi) y(\xi, \varepsilon) + w(\xi), \quad \xi \in (-1, 1), \quad (20)$$

$$\begin{aligned} \langle T^2 y(-1, \varepsilon), N(-1, \varepsilon^{-1} S) \rangle &= \sigma_1, & \langle T y(-1, \varepsilon), N(-1, \varepsilon^{-1} S) \rangle &= \sigma_2, \\ \langle T^2 y(1, \varepsilon), N(1, \varepsilon^{-1} S) \rangle &= \sigma_3, & \langle T y(1, \varepsilon), N(1, \varepsilon^{-1} S) \rangle &= \sigma_4, \end{aligned} \quad (21)$$

has a unique solution $y(\cdot, \varepsilon_l)$ for each $l = 1, 2, \dots$.

The family of solutions $\{y(\cdot, \varepsilon_l)\}_{l=1}^\infty$ satisfies the estimate

$$\|y(\cdot, \varepsilon_l) - y_*\|_{C^1} \leq C e^{-\varepsilon_p^{-1} M}$$

with y_* being a smooth vector function on $[-1, 1]$, and with positive constants C and M that are independent of ε .

P r o o f. System (20) has a fundamental matrix

$$\begin{aligned} \Phi &= q^{-3/8} \begin{pmatrix} \Phi_1 & 0 \\ 0 & \Phi_2 \end{pmatrix}, \quad \Phi_1 = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix}, \\ \Phi_2 &= \begin{pmatrix} e^{\alpha(-1)-\alpha} & 0 \\ 0 & e^{\alpha-\alpha(1)} \end{pmatrix} \end{aligned}$$

with a function

$$\alpha(\xi) = \frac{1}{4} (\lambda_0^{-3} k_0^{-5}(0))^{1/4} \int_{-1}^{\xi} q^{1/4}(t) (\lambda_1 k_0(0) - \lambda_0 k_0'(0)t) dt.$$

Then the general solution of (20) can be written as

$$y(\xi) = \Phi(\xi)(\beta + h(\xi)),$$

where β is a constant vector, and

$$h(\xi) = \int_{-1}^{\xi} \Phi^{-1}(t) w(t) dt.$$

Let further

$$y(\xi, \varepsilon) = \Phi(\xi)(\beta_\varepsilon + h(\xi))$$

be a solution of boundary value problem (20), (21). It is easy to check that

$$\Phi^t(\xi) N(\xi, \varepsilon^{-1} S) = q^{-3/8}(\xi) N(\xi, \gamma_\varepsilon), \quad \text{where } \gamma_\varepsilon(\xi) = \varepsilon^{-1} S(\xi) + \alpha(\xi),$$

and Φ^t is transposed to Φ . Therefore we get

$$\langle y(\xi, \varepsilon), N(\xi, \varepsilon^{-1} S) \rangle = q^{-3/8}(\xi) \langle \beta_\varepsilon + h(\xi), N(\xi, \gamma_\varepsilon) \rangle,$$

that allows to rewrite boundary conditions (21) to the form

$$\begin{aligned} \langle \beta_\varepsilon, T^2 N(-1, \gamma_\varepsilon) \rangle &= m^- \sigma_1, \\ \langle \beta_\varepsilon, T^3 N(-1, \gamma_\varepsilon) \rangle &= m^- \sigma_2, \\ \langle \beta_\varepsilon, T^2 N(1, \gamma_\varepsilon) \rangle &= m^+ \sigma_3 - \langle h(1), T^2 N(1, \gamma_\varepsilon) \rangle, \\ \langle \beta_\varepsilon, T^3 N(1, \gamma_\varepsilon) \rangle &= m^+ \sigma_4 - \langle h(1), T^3 N(1, \gamma_\varepsilon) \rangle, \end{aligned} \quad (22)$$

where $m^\pm = q^{3/8}(\pm 1)$. Here we have used the equalities $T\Phi^t = \Phi^t T$ and $h(-1) = 0$. Therefore, the vector β_ε has to be a solution to a linear algebraic system with matrix

$$G(\gamma_\varepsilon(1)) = \begin{pmatrix} -1 & 0 & 1 & e^{-\gamma_\varepsilon(1)} \\ 0 & -1 & -1 & e^{-\gamma_\varepsilon(1)} \\ -\cos \gamma_\varepsilon(1) & -\sin \gamma_\varepsilon(1) & e^{-\gamma_\varepsilon(1)} & 1 \\ \sin \gamma_\varepsilon(1) & -\cos \gamma_\varepsilon(1) & -e^{-\gamma_\varepsilon(1)} & 1 \end{pmatrix}.$$

Note that the determinant

$$\det G(\gamma_\varepsilon(1)) = -2 \cos \gamma_\varepsilon(1) + 2e^{-\gamma_\varepsilon(1)} (2 - e^{-\gamma_\varepsilon(1)} \cos \gamma_\varepsilon(1))$$

becomes zero on an infinitely small sequence ε (but not for any $\varepsilon > 0$ since $\gamma_\varepsilon(1) \rightarrow \infty$ as $\varepsilon \rightarrow 0$).

We fix a number δ from the interval $[0, 2\pi)$ such that δ is different from $\pi/2$ and $3\pi/2$. We construct the sequence ε_l using the set of conditions $\gamma_{\varepsilon_l}(1) = \delta + 2\pi l$ for $l = 1, 2, \dots$. In other words,

$$\varepsilon_l(\delta) = \frac{S(1)}{\delta + 2\pi l - \alpha(1)} \quad (23)$$

for all $l \geq l_0$, where l_0 is the smallest natural number such that the denominator of fraction (23) is positive. We use notation $\gamma_l = \gamma_{\varepsilon_l}$. Using $\gamma_l \rightarrow \infty$ as $\varepsilon_l \rightarrow 0$, we can treat the matrix $G(\gamma_l)$ as an exponentially small perturbation of matrix

$$G_\delta = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ -\cos \delta & -\sin \delta & 0 & 1 \\ \sin \delta & -\cos \delta & 0 & 1 \end{pmatrix}.$$

In the same manner $N(1, \gamma_l) = (\cos \delta, \sin \delta, e^{-\gamma_l(1)}, 1)$, then the right hand side of system (22) also is an exponentially small perturbation of the vector

$$g = (m^- \sigma_1, m^- \sigma_2, m^+ \sigma_3 - \langle h(1), T^2 N_1 \rangle, m^+ \sigma_4 - \langle h(1), T^3 N_1 \rangle),$$

where N_1 differs from $N(1, \gamma_l)$ only by the third coordinate being zero.

For the chosen value δ the matrix G_δ is a non-degenerate one. Then according to the results of perturbation theory in a finite dimensional space, we have

$$\|\beta_{\varepsilon_l} - \beta_*\|_{\mathbb{R}^4} \leq C e^{-\gamma_l(1)},$$

where β_* is a solution to the system $G_\delta \beta = g$. Let $y_*(\xi) = \Phi(\xi)(\beta_* + h(\xi))$, then

$$\|y(\cdot, \varepsilon_l) - y_*\|_{C^1} \leq \|\Phi\|_{C^1} \|\beta_{\varepsilon_l} - \beta_*\|_{\mathbb{R}^4},$$

with $\|\Phi\|_{C^1} = \max_{\xi \in [-1, 1]} (\|\Phi(\xi)\| + \|\Phi'(\xi)\|)$. To finish the proof, it remains to note that $\gamma_\varepsilon(1) \geq M\varepsilon^{-1}$ with a certain positive constant M . \square

Remark. The function y_* is referred to as the principal solution of problem (20), (21) since constructing asymptotic expansions in power scale of ε we can neglect exponentially small terms. Nevertheless, the choice of the sequence ε_l , and then the choice of the solution y_* , is not unique since it depends on δ . As we mentioned at the beginning, the latter is connected to the fact that the discrete approximation of global vibrations allows deformation, hence the approximation is not unique. A presence of the deformation parameter δ in asymptotics corresponds to the problem content.

We come back to exploring problem (18), (19), which is problem (20), (21) with the right-hand side $w = 0$ and $\sigma = (S'(-1)^{-2} v_0''(-0), 0, 0, 0)$. The system is homogeneous, and then $f_0(\xi) = \Phi(\xi)\beta_0$, where

$$\beta_0 = 1/2 \, q^{3/8} (-1) S'(-1)^{-2} v_0''(-0) (\operatorname{tg} \delta - 1, -\operatorname{tg} \delta - 1, \operatorname{tg} \delta + 1, -1/\cos \delta)$$

is a solution of the corresponding linear system with matrix G_δ .

Hence, we have found the terms $\lambda_0, v_0, \lambda_1, v_1, f_0$ of expansions (5), (6), (9). We recall that f_0 depends on value δ . The construction of complete asymptotic expansions for solutions to problem (1) – (4) is conducted on the sequence $\mathcal{E}_\delta = \{\varepsilon_l(\delta)\}_{l_0}^\infty$ that is chosen accordingly to (23).

3. Complete asymptotics of global vibrations. Let find coefficients λ_i, v_i, f_{i-1} of asymptotic expansions (5), (6), (9) for $i \geq 2$. Using conditions (11), (13), (14) we

construct a boundary value problem for the general term v_i in the form

$$\begin{aligned} Lv_i - \lambda_0 p(x) v_i &= p(x) \sum_{j=1}^i \lambda_j v_{i-j}, \quad x \in (a, 0) \cup (0, b), \\ v_i(a) &= v'_i(a) = 0, \quad v_i(b) = v'_i(b) = 0, \\ v_i(\pm 0) &= V_i(\pm 0), \quad v'_i(\pm 0) = W_i(\pm 0). \end{aligned} \quad (24)$$

The right-hand sides of the boundary conditions in problem (24) with the precision up to exponentially small terms are

$$V_i(\pm 0) = \langle q^{-3/8} \Phi^{-1} f_{i-4}(\pm 1), N_{\pm 1} \rangle - \sum_{j=0}^{i-1} (\pm 1)^j (j!)^{-1} v_{i-j}^{(j)}(\pm 0),$$

and

$$W_i(\pm 0) = \langle q^{-3/8} \Phi^{-1} (S' T^3 f_{i-2} + f'_{i-3})(\pm 1), N_{\pm 1} \rangle - \sum_{j=0}^{i-1} (\pm 1)^j (j!)^{-1} v_{i-j}^{(j+1)}(\pm 0),$$

where $N_{-1} = (1, 0, 1, 0)$, and the vector N_1 was introduced during the proof of Proposition 1. Since λ_0 is an eigenvalue of problem (16), boundary value problem (24) could have no solution. The necessary and sufficient condition for existence of the solution is equality $\lambda_i = (k_0 v_0'' W_i - (k_0 v_0'')' V_i)(-0)$. We fix the solution v_i to problem (24) such that

$$\int_a^0 v_i v_0 = 0.$$

Then we find f_{i-1} from the problem

$$\begin{aligned} f'_i &= A(\xi) f_i + 1/4 k_0^{-1}(0) S'^{-3} T^3 \chi_{i+1}, \quad \xi \in (-1, 1), \\ \langle T^2 f_i(\pm 1), N(\pm 1, \varepsilon^{-1} S) \rangle &= -\langle (q^{-3/8} S'^{-2} \Phi^{-1} D_i)(\pm 1), N_{\pm 1} \rangle + \\ &S'(\pm 1)^{-2} \sum_{j=0}^i (\pm 1)^j (j!)^{-1} v_{i-j}^{j+2}(\pm 0), \\ \langle T f_i(\pm 1), N(\pm 1, \varepsilon^{-1} S) \rangle &= \\ &S'(\pm 1)^{-3} \left(F_i(\pm 0) - \langle (q^{-3/8} \Phi^{-1} E_i)(\pm 1), N_{\pm 1} \rangle \right). \end{aligned} \quad (25)$$

According to Proposition 1, the right-hand side of the system is a smooth function. Then for the sequence of small parameter \mathcal{E}_δ there is a solution to problem (25).

Hence the algorithm of constructing coefficients of expansions (5), (6), (9) and a sequence of small parameter is

$$\lambda_0 \rightarrow v_0 \rightarrow \lambda_1 \rightarrow v_1 \rightarrow \mathcal{E}_\delta \rightarrow f_0 \rightarrow \dots \rightarrow \lambda_i \rightarrow v_i \rightarrow f_{i-1} \dots$$

We recall that all the coefficients, except λ_0 , v_0 and λ_1 , v_1 , depend on the value of parameter δ . In other words, to each $\delta \in [0, 2\pi)$, $\delta \neq \pi/2, 3\pi/2$ we assign the corresponding asymptotic series that approximate the same form of global vibrations v_0 .

4. Justification of asymptotics. The idea of justification is, using asymptotic series (5), to choose exactly the eigenfunction sequence of problem (1) – (4) such that the sequence models global vibrations and is approximated by series (6), (9).

We fix numbers $n \in \mathbb{N}$ and $\delta \in [0, 2\pi)$ different from $\pi/2$ and $3\pi/2$. We introduce a number sequence $\{\lambda_\varepsilon^{\{n\}}\}_{\varepsilon \in \mathcal{E}_\delta}$ and a sequence of functions $\{u_\varepsilon^{\{n\}}\}_{\varepsilon \in \mathcal{E}_\delta}$. Namely, for each element ε of the discrete set \mathcal{E}_δ we assume

$$\lambda_\varepsilon^{\{n\}} = \lambda_0 + \varepsilon \lambda_1 + \dots + \varepsilon^n \lambda_n,$$

$$u_\varepsilon^{\{n\}}(x) = \begin{cases} v_0(x) + \varepsilon v_1(x) + \cdots + \varepsilon^n v_n(x), & x \in (a, -\varepsilon) \cup (\varepsilon, b), \\ \varepsilon^4 \sum_{i=0}^{n+2} \varepsilon^i \langle f_i(\varepsilon^{-1}x), N(\varepsilon^{-1}x, \varepsilon^{-1}S) \rangle, & x \in (-\varepsilon, \varepsilon), \end{cases}$$

where the values λ_i , functions v_i , vectors f_i and the set \mathcal{E}_δ are constructed in Sections 2 and 3.

Proposition 2. *There exists an eigenvalue sequence $\{\lambda^\varepsilon\}_\varepsilon \in \mathcal{E}_\delta$ such that*

$$|\lambda_\varepsilon^{\{n\}} - \lambda^\varepsilon| \leq C_n \varepsilon^{n+1}, \quad \varepsilon \in \mathcal{E}_\delta.$$

P r o o f. Denote by ρ_ε the density of the original problem, which equals $\varepsilon^{-8}q(x/\varepsilon)$ on the interval $(-\varepsilon, \varepsilon)$, and is $p(x)$ outside the interval. Problem (1) – (4) considered in the weight space $L_2(\rho_\varepsilon, (a, b))$ is equivalent to the problem for the self-adjoint operator $\mathcal{L}_\varepsilon = \rho_\varepsilon^{-1}L$:

$$\mathcal{L}_\varepsilon u_\varepsilon - \lambda_\varepsilon u_\varepsilon = 0.$$

The domain of the operator \mathcal{L}_ε is

$$D(\mathcal{L}_\varepsilon) = \{\varphi \in H^4(a, b) : \varphi(a) = \varphi'(a) = 0, \quad \varphi(b) = \varphi'(b) = 0\}.$$

The spectrum of the operator \mathcal{L}_ε is discrete [1].

The function $u_\varepsilon^{\{n\}}$ is not an element of the space $D(\mathcal{L}_\varepsilon)$ since it has discontinuities at the points $x = \pm\varepsilon$. Nevertheless, we state existence of the function $\varphi_\varepsilon^{\{n\}}$ such that $u_\varepsilon^{\{n\}} + \varphi_\varepsilon^{\{n\}} \in D(\mathcal{L}_\varepsilon)$, moreover $\varphi_\varepsilon^{\{n\}}$ is equal to zero on the interval $(-\varepsilon, \varepsilon)$, and

$$\max_{x \in (a, b)} \sum_{i=0}^4 \left| \frac{d^i}{dx^i} \varphi_\varepsilon^{\{n\}} \right| \leq C_n \varepsilon^{n+1}. \quad (26)$$

We introduce function

$$V_\varepsilon^{\{n\}} = \kappa_\varepsilon (u_\varepsilon^{\{n\}} + \varphi_\varepsilon^{\{n\}}),$$

where the factor κ_ε is chosen accordingly to

$$\|V_\varepsilon^{\{n\}}\|_{L_2(\rho_\varepsilon, (a, b))} = 1.$$

We remark that the value κ_ε is separated from zero by a positive constant that does not depend on ε .

Taking into account (26), we can show that

$$\left\| \mathcal{L}_\varepsilon V_\varepsilon^{\{n\}} - \lambda_\varepsilon^{\{n\}} V_\varepsilon^{\{n\}} \right\|_{L_2(\rho_\varepsilon, (a, b))} \leq K_n \varepsilon^{n+1}, \quad \varepsilon \in \mathcal{E}_\delta,$$

where the constant K_n is independent of ε . Therefore there exists an eigenvalue λ^ε of the operator \mathcal{L}_ε such that

$$\left| \lambda^\varepsilon - \lambda_\varepsilon^{\{n\}} \right| \leq K_n \varepsilon^{n+1}. \quad \square$$

Using asymptotics of each eigenvalue λ_n^ε we can prove that in a certain $d\varepsilon^4$ -vicinity of points λ_ε , which were chosen accordingly to Proposal 2, we don't have other eigenvalues of the initial problem. As consequence, for $n \geq 5$ we have only one eigenvalue λ_ε that satisfies Proposition 2. We choose the sequence of exactly these $\{\lambda_\varepsilon\}_\varepsilon \in \mathcal{E}_\delta$ and the corresponding normalized eigenfunctions $\{u_\varepsilon\}_\varepsilon \in \mathcal{E}_\delta$. Then

$$\|u_\varepsilon - \kappa_\varepsilon v_\varepsilon^{\{n\}}\|_{L_2(\rho_\varepsilon, (a, b))} \leq K_n d^{-1} \varepsilon^{n-3}, \quad \|u_\varepsilon\|_{L_2(\rho_\varepsilon, (a, b))} = 1.$$

This inequality complete the proof of

Theorem. *On the sequence \mathcal{E}_δ the chosen functions u_ε satisfy the estimates*

$$\left\| u_\varepsilon(x) - \kappa_\varepsilon \sum_{k=0}^n \varepsilon^k v_k(x) \right\|_{L_2(\Omega_\varepsilon)} \leq C_n \varepsilon^{n+1},$$

where region Ω_ε states for $(a, -\varepsilon)$ or (ε, b) , and the estimate

$$\left\| u_\varepsilon(\varepsilon\xi) - \varepsilon^4 \kappa_\varepsilon \sum_{k=0}^n \varepsilon^k \langle f_k(\xi), N(\xi, \varepsilon^{-1}S) \rangle \right\|_{L_2(-1,1)} \leq C_n \varepsilon^{n+1}$$

for $n = 0, 1, \dots$. The coefficients v_k , f_k are the same as at the beginning of Section 4. The normalizing multiplier κ_ε tends to 1 as $\varepsilon \rightarrow 0$.

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